The Barut Second-Order Equation, Dynamical Invariants and Interactions

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Abstract

The second-order equation in the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group has been proposed by A. Barut in the beginning of the 70s, ref. [1]. It permits to explain the mass splitting of leptons (e,μ,τ). Recently, the interest has grown to this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier *et al.* [3]). We continue the research deriving the equation from the first principles, finding the dynamical invariants for this model, investigating the influence of the potential interactions.

The Barut main equation is

$$[i\gamma_\mu\partial_\mu - \alpha_2\frac{\partial_\mu\partial_\mu}{m} + \kappa]\Psi = 0\,. \tag{1}$$

- It represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the $O(4,2)$ group, $N_{ab} = \frac{i}{2}\gamma_a\gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5, i\}$.

- Instead of 4 solutions it has 8 solutions with the correct relativistic relation $E = \pm\sqrt{\mathbf{p}^2 + m_i^2}$. In fact, it describes states of different masses (the second one is $m_\mu = m_e(1+\frac{3}{2\alpha})$, α is the fine structure constant), provided that a certain physical condition is imposed on the α_2 parameter (the anomalous magentic moment should be equal to $4\alpha/3$).

- One can also generalize the formalism to include the third state, the τ-lepton [1b].

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- Barut has indicated at the possibility of including γ_5 terms (e.g., $\sim \gamma_5\kappa^{'}$).

If we present the 4-spinor as $\Psi(\mathbf{p}) = column(\phi_R(\mathbf{p})) \quad \phi_L(\mathbf{p}))$ then Ryder states [5] that $\phi_R(\mathbf{0}) = \phi_L(\mathbf{0})$. Similar argument has been given by Faustov [6]: "the matrix B exists such that $Bu_\lambda(\mathbf{0}) = u_\lambda(\mathbf{0})$, $B^2 = I$ for any $(2J+1)$-component function within the Lorentz invariant theories"[1] The most general form of the relation in the $(1/2,0)\oplus(0,1/2)$ representation has been given by Dvoeglazov [7,4a]:

$$\phi_L^h(\mathbf{0}) = a(-1)^{\frac{1}{2}-h} e^{i(\theta_1+\theta_2)}\Theta_{1/2}[\phi_L^{-h}(\mathbf{0})]^* + be^{2i\theta_h}\Xi_{1/2}^{-1}[\phi_L^h(\mathbf{0})]^*\,, \tag{2}$$

with

$$\Theta_{1/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2\,, \quad \Xi_{1/2} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\,, \tag{3}$$

Θ_J is the Wigner operator for spin $J = 1/2$, φ is the azimuthal angle $\mathbf{p}\rightarrow 0$ of the spherical coordinate system.

Next, we use the Lorentz transformations:

$$\Lambda_{R,L} = \exp(\pm\sigma\cdot\phi/2),\ cosh\,\phi = E_p/m,\ sinh\,\phi = |\mathbf{p}|/m,\ \hat{\phi} = \mathbf{p}/|\mathbf{p}|. \tag{4}$$

Applying the boosts and the relations between spinors in the rest frame, one can obtain:

$$\phi_L^h(\mathbf{p}) \quad = \quad a\frac{p_0 - \sigma\cdot\mathbf{p}}{m}\phi_R^h(\mathbf{p}) + b(-1)^{\frac{1}{2}+h}\Theta_{1/2}\Xi_{1/2}\phi_R^{-h}(\mathbf{p}), \tag{5}$$

$$\phi_R^h(\mathbf{p}) \quad = \quad a\frac{p_0 + \sigma\cdot\mathbf{p}}{m}\phi_L^h(\mathbf{p}) + b(-1)^{\frac{1}{2}+h}\Theta_{1/2}\Xi_{1/2}\phi_L^{-h}(\mathbf{p}). \tag{6}$$

($\theta_1 = \theta_2 = 0$, $p_0 = E_p = \sqrt{\mathbf{p}^2+m^2}$). In the Dirac form we have:

$$[a\frac{\hat{p}}{m} - 1]u_h(\mathbf{p}) + ib(-1)^{\frac{1}{2}-h}\gamma^5\mathcal{C}u^*_{-h}(\mathbf{p}) = 0, \tag{7}$$

where $\mathcal{C} = \begin{pmatrix} 0 & i\Theta_{1/2} \\ -i\Theta_{1/2} & 0 \end{pmatrix}$. In the QFT form we must introduce the creation/annihilation operators. Let $b_\downarrow = -ia_\uparrow$, $b_\uparrow = +ia_\downarrow$, then

$$[a\frac{i\gamma^\mu\partial_\mu}{m} + b\mathcal{C}\mathcal{K} - 1]\Psi(x^\nu) = 0\,. \tag{8}$$

If one applies the unitary transformation to the Majorana representation [8]

$$\mathcal{U} = \frac{1}{2}\begin{pmatrix} 1 - i\Theta_{1/2} & 1 + i\Theta_{1/2} \\ -1 - i\Theta_{1/2} & 1 - i\Theta_{1/2} \end{pmatrix}\,,\ \mathcal{U}\mathcal{C}\mathcal{K}\mathcal{U}^{-1} = -\mathcal{K}\,, \tag{9}$$

[1] The latter statement is more general than the Ryder one, because it admits $B = \begin{pmatrix} 0 & e^{+i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}$, so that $\phi_R(\mathbf{0}) = e^{i\alpha}\phi_L(\mathbf{0})$.

then γ-matrices become to be pure imaginary, and the equations are pure real.

$$\left[a\frac{i\hat{\partial}}{m} - b - 1\right]\Psi_1 \quad = \quad 0\,, \qquad (10)$$

$$\left[a\frac{i\hat{\partial}}{m} + b - 1\right]\Psi_2 \quad = \quad 0\,, \qquad (11)$$

where $\Psi = \Psi_1 + i\Psi_2$. It appears as if the real and imaginary parts have different masses. Finally, for superpositions $\phi = \Psi_1 + \Psi_2$, $\chi = \Psi_1 - \Psi_2$, multiplying by $b \neq 0$ we have:

$$[2a\frac{i\gamma^\mu\partial_\mu}{m} + a^2\frac{\partial^\mu\partial_\mu}{m^2} + b^2 - 1]\frac{\phi(x^\nu)}{\chi(x^\nu)} = 0\,, \qquad (12)$$

If we put $a/2m \to \alpha_2$, $\frac{1-b^2}{2a}m \to \kappa$ we recover the Barut equation.

How can we get the third lepton state? See the refs. [1b,4b]:

$$M_\tau = M_\mu + \frac{3}{2}\alpha^{-1}n^4 M_e = M_e + \frac{3}{2}\alpha^{-1}1^4 M_e + \frac{3}{2}\alpha^{-1}2^4 M_e = 1786.08\,MeV\,. \quad (13)$$

The physical origin was claimed by Barut to be in the magnetic self-interaction of the electron (the factor n^4 appears due to the Bohr-Sommerfeld rule for the charge moving in circular orbits in the field of a fixed magnetic dipole μ). One can start from (7), but , as opposed to the above-mentioned, one can write the coordinate-space equation in the form:

$$[a\frac{i\gamma^\mu\partial_\mu}{m} + b_1\mathcal{C}\mathcal{K} - 1]\Psi(x^\nu) + b_2\gamma^5\mathcal{C}\mathcal{K}\tilde{\Psi}(x^\nu) = 0\,, \qquad (14)$$

with $\Psi^{MR} = \Psi_1 + i\Psi_2$, $\tilde{\Psi}^{MR} = \Psi_3 + i\Psi_4$. As a result,

$$(a\frac{i\gamma^\mu\partial_\mu}{m} - 1)\phi - b_1\chi + ib_2\gamma^5\tilde{\phi} \quad = \quad 0\,, \qquad (15)$$

$$(a\frac{i\gamma^\mu\partial_\mu}{m} - 1)\chi - b_1\phi - ib_2\gamma^5\tilde{\chi} \quad = \quad 0\,. \qquad (16)$$

The operator $\tilde{\Psi}$ may be linear-dependent on the states included in the Ψ. let us apply the most simple form $\Psi_1 = -i\gamma^5\Psi_4$, $\Psi_2 = +i\gamma^5\Psi_3$. Then, one can recover the 3rd order Barut-like equation [4b]:

$$[i\gamma^\mu\partial_\mu - m\frac{1 \pm b_1 \pm b_2}{a}][i\gamma^\nu\partial_\nu + \frac{a}{2m}\partial^\nu\partial_\nu + m\frac{b_1^2 - 1}{2a}]\Psi_{1,2} = 0\,. \qquad (17)$$

Thus, we have three mass states.

Let us reveal the connections with other models. For instance, in refs. [3, 9] the following equation has been studied:

$$[(i\hat{\partial} - e\hat{A})(i\hat{\partial} - e\hat{A}) - m^2]\Psi = [(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} - m^2]\Psi = 0 \qquad (18)$$

for the 4-component spinor Ψ. This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of ref. [3c]):

$$\mathcal{L}_0 = (i\overline{\hat{\partial}\Psi})(i\hat{\partial}\Psi) - m^2\bar{\Psi}\Psi\,. \tag{19}$$

We can note:

- The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of Eq. (19) with the dark matter [10], provided that Ψ is composed of the self/anti-self charge conjugate spinors, and it has the dimension $[energy]^1$ in $c = \hbar = 1$. The interaction Lagrangian is $\mathcal{L}^H \sim g\bar{\Psi}\Psi\phi^2$.
- The term $\sim \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$ will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order $\sim e^2$) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of γ^5 operator.
- In general, J_0 is not the positive-defined quantity, since the general solution $\Psi = a\Psi_+ + b\Psi_-$, where $[i\gamma^\mu\partial_\mu \pm m]\Psi_\pm = 0$, see also [11].

The most general conserved current of the Barut-like theories is

$$J_\mu = \alpha_1\gamma_\mu + \alpha_2 p_\mu + \alpha_3\sigma_{\mu\nu}q^\nu\,. \tag{20}$$

Let us try the Lagrangian:

$$\mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{add}\,, \tag{21}$$

$$\mathcal{L}_{Dirac} = \alpha_1[\bar{\Psi}\gamma^\mu(\partial_\mu\Psi) - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi] - \alpha_4\bar{\Psi}\Psi\,, \tag{22}$$

$$\mathcal{L}_{add} = \alpha_2(\partial_\mu\bar{\Psi})(\partial^\mu\Psi) + \alpha_3\partial_\mu\bar{\Psi}\sigma^{\mu\nu}\partial_\nu\Psi\,. \tag{23}$$

Then, the equation follows:

$$[2\alpha_1\gamma^\mu\partial_\mu - \alpha_2\partial_\mu\partial^\mu - \alpha_4]\Psi = 0\,, \tag{24}$$

and its Dirac-conjugate:

$$\bar{\Psi}[2\alpha_1\gamma^\mu\partial_\mu + \alpha_2\partial_\mu\partial^\mu + \alpha_4] = 0\,. \tag{25}$$

The derivatives acts to the left in the second equation. Thus, we have the Dirac equation when $\alpha_1 = \frac{i}{2}$, $\alpha_2 = 0$, and the Barut equation when $\alpha_2 = \frac{1}{m}\frac{2\alpha/3}{1+4\alpha/3}$.

In the Euclidean metrics the dynamical invariants are

$$\mathcal{J}_\mu = -i\sum_i [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_i)}\Psi_i - \bar\Psi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\Psi_i)}]\,, \tag{26}$$

$$\mathcal{T}_{\mu\nu} = -\sum_i [\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_i)}\partial_\nu \Psi_i + \partial_\nu \bar\Psi_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\Psi_i)}] + \mathcal{L}\delta_{\mu\nu}\,, \tag{27}$$

$$\mathcal{S}_{\mu\nu,\lambda} = -i\sum_{ij} [\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Psi_i)} N^{\Psi}_{\mu\nu,ij}\Psi_j + \bar\Psi_i N^{\bar\Psi}_{\mu\nu,ij}\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \bar\Psi_j)}]\,. \tag{28}$$

$N^{\Psi,\bar\Psi}_{\mu\nu}$ are the Lorentz group generators.

Then, the energy-momentum tensor is

$$\begin{aligned}\mathcal{T}_{\mu\nu} &= -\alpha_1[\bar\Psi\gamma_\mu\partial_\nu\Psi - \partial_\nu\bar\Psi\gamma_\mu\Psi] - \alpha_2[\partial_\mu\bar\Psi\partial_\nu\Psi + \partial_\nu\bar\Psi\partial_\mu\Psi] - \alpha_3\left[\partial_\alpha\bar\Psi\sigma_{\alpha\mu}\partial_\nu\Psi + \right.\\ &+ \left.\partial_\nu\bar\Psi\sigma_{\mu\alpha}\partial_\alpha\Psi\right] + \left[\alpha_1(\bar\Psi\gamma_\mu\partial_\mu\Psi - \partial_\mu\bar\Psi\gamma_\mu\Psi) + \alpha_2\partial_\alpha\bar\Psi\partial_\alpha\Psi + \alpha_3\partial_\alpha\bar\Psi\sigma_{\alpha\beta}\partial_\beta\Psi + \right.\\ &+ \left.\alpha_4\bar\Psi\Psi\right]\delta_{\mu\nu}\,.\end{aligned} \tag{29}$$

Hence, the Hamiltonian $\hat{\mathcal{H}} = -iP_4 = -\int \mathcal{T}_{44}d^3x$ is

$$\hat{\mathcal{H}} = \int\{\alpha_1[\partial_i\bar\Psi\gamma_i\Psi - \bar\Psi\gamma_i\partial_i\Psi] + \alpha_2[\partial_4\bar\Psi\partial_4\Psi - \partial_i\bar\Psi\partial_i\Psi] - \alpha_3[\partial_i\bar\Psi\sigma_{ij}\partial_j\Psi] - \alpha_4\bar\Psi\Psi\}d^3x\,. \tag{30}$$

The 4-current is

$$\mathcal{J}_\mu = -i\{2\alpha_1\bar\Psi\gamma_\mu\Psi + \alpha_2[(\partial_\mu\bar\Psi)\Psi - \bar\Psi(\partial_\mu\Psi)] + \alpha_3[\partial_\alpha\bar\Psi\sigma_{\alpha\mu}\Psi - \bar\Psi\sigma_{\mu\alpha}\partial_\alpha\Psi]\}\,. \tag{31}$$

Hence, the charge operator $\hat{\mathcal{Q}} = -i\int \mathcal{J}_4 d^3x$ is

$$\hat{\mathcal{Q}} = -\int\{2\alpha_1\Psi^\dagger\Psi + \alpha_2[(\partial_4\bar\Psi)\Psi - \bar\Psi(\partial_4\Psi)] + \alpha_3[\partial_i\bar\Psi\sigma_{i4}\Psi - \bar\Psi\sigma_{4i}\partial_i\Psi]\}d^3x\,. \tag{32}$$

Finally, the spin tensor is

$$\begin{aligned}\mathcal{S}_{\mu\nu,\lambda} &= -\frac{i}{2}\left\{\alpha_1[\bar\Psi\gamma_\lambda\sigma_{\mu\nu}\Psi + \bar\Psi\sigma_{\mu\nu}\gamma_\lambda\Psi] + \alpha_2[\partial_\lambda\bar\Psi\sigma_{\mu\nu}\Psi - \bar\Psi\sigma_{\mu\nu}\partial_\lambda\Psi] + \right. \\ &\left. +\alpha_3[\partial_\alpha\bar\Psi\sigma_{\alpha\lambda}\sigma_{\mu\nu}\Psi - \bar\Psi\sigma_{\mu\nu}\sigma_{\lambda\alpha}\partial_\alpha\Psi]\right\}\,.\end{aligned} \tag{33}$$

In the quantum case the corresponding field operators are written:

$$\Psi(x^m u) = \sum_h \int \frac{d^3\mathbf{p}}{(2\pi)^3}[u_h(\mathbf{p})a_h(\mathbf{p})e^{+ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})e^{-ip\cdot x}]\,, \tag{34}$$

$$\bar\Psi(x^m u) = \sum_h \int \frac{d^3\mathbf{p}}{(2\pi)^3}[\bar u_h(\mathbf{p})a_h^\dagger(\mathbf{p})e^{-ip\cdot x} + \bar v_h(\mathbf{p})b_h(\mathbf{p})e^{+ip\cdot x}]\,. \tag{35}$$

The 4-spinor normalization is

$$\bar{u}_h u_{h'} = \delta_{hh'}\,, \quad \bar{v}_h v_{h'} = -\delta_{hh'}\,. \tag{36}$$

The commutation relations are

$$\left[a_h(\mathbf{p}), a^\dagger_{h'}(\mathbf{k})\right]_+ = (2\pi)^3 \frac{m}{p_4}\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{hh'}\,, \tag{37}$$

$$\left[b_h(\mathbf{p}), b^\dagger_{h'}(\mathbf{k})\right]_+ = (2\pi)^3 \frac{m}{p_4}\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{hh'}\,, \tag{38}$$

with all other to be equal to zero. The dimensions of the Ψ, $\bar{\Psi}$ are as usual, $[energy]^{3/2}$. Hence, the second-quantized Hamiltonian is written

$$\hat{\mathcal{H}} = -\sum_h \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{2E_p^2}{m}[\alpha_1 + m\alpha_2] : [a^\dagger_h a_h - b_h b^\dagger_h] : \,. \tag{39}$$

(Remember that $\alpha_1 \sim \frac{i}{2}$, the commutation relations may give another i, so the contribution of the first term to eigenvalues will be real. But if α_2 is real, the contribution of the second term may be imaginary). The charge is

$$\hat{\mathcal{Q}} = -\sum_{hh'} \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{2E_p}{m}[(\alpha_1 + m\alpha_2)\delta_{hh'} - i\alpha_3 \bar{u}_h \sigma_{i4} p_i u_{h'}] : [a^\dagger_h a_{h'} + b_h b^\dagger_{h'}] : \,. \tag{40}$$

However, due to $[\Lambda_{R,L}, \sigma\cdot\mathbf{p}]_- = 0$ the last term with α_3 does not contribute.

The conclusions are:

- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives. The Majorana representation has been used.
- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
- We found relations with other models (such as the Feynman-Gell-Mann equation).
- As a result of analysis of dynamical invariants, we can state that at the free level the term $\sim \alpha_3 \partial_\mu \bar{\Psi}\sigma_{\mu\nu}\partial_\nu \Psi$ in the Lagrangian does not contribute.
- However, the interaction terms $\sim \alpha_3 \bar{\Psi}\sigma_{\mu\nu}\partial_\nu \Psi A_\mu$ will contribute when we construct the Feynman diagrams and the S-matrix. In the curved space (the 4-momentum Lobachevsky space) the influence of such terms has been investigated in the Skachkov works [12]. Briefly, the contribution will be such as if the 4-potential were interact with some "renormalized" spin. Perhaps, this explains, why did Barut use the classical anomalous magnetic moment $g \sim 4\alpha/3$ instead of $\frac{\alpha}{2\pi}$.

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